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FACTOR ORBIT EQUIVALENCE OF COMPACT GROUP EXTENSIONS

BY

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ABSTRACT

Let G be a compact metrizable group. We show that any two ergodic extensions of transformations T_1 and T_2 by rotations of G are factor orbit equivalent relative to T_1 and T_2, and the equivalence may be taken to have a certain natural form.

1. Introduction

Throughout this paper, all transformations are assumed to be (or must be shown to be) measure-preserving automorphisms of Lebesgue probability spaces. Transformations T_i on spaces $(X_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$, with factors $\mathcal{A}_i$ are said to be factor orbit equivalent (relative to $\mathcal{A}_1$ and $\mathcal{A}_2$) if there exists an orbit equivalence $\Phi: X_1 \to X_2$ between T_1 and T_2 such that $\Phi(\mathcal{A}_1) = \mathcal{A}_2$. In [3], M. Gerber gave a classification up to factor orbit equivalence of the finite extensions of an ergodic transformation. In particular, she obtained the somewhat surprising result that this classification is non-trivial; two ergodic extensions by discrete spaces of the same cardinality need not be factor orbit equivalent. Furthermore, she showed that ergodic extensions of a transformation by non-atomic spaces also need not be factor orbit equivalent. In this paper we obtain the positive result that the ergodic extensions of a transformation by rotations of a compact metrizable group form a single factor orbit equivalence class. We remark that the special case of this result, in which the group in question is finite, follows from the classification of [3], and that the positive portion of that classification (that finite extensions with the G-interchange property are factor orbit equivalent) can be seen to follow from our result.

To formulate our result more precisely, let $(G, \mathcal{B}, \lambda, \rho)$ denote a compact metric group (which will be fixed for the remainder of the paper) with Borel sets $\mathcal{B}$, Haar measure λ, and invariant metric $\rho \leqq 1$.

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Given a transformation T on $(X, \mathscr{A}, \mu)$ and a measurable function $\sigma : X \to G$, we form a transformation S on $(X \times G, \mathscr{A} \times \mathscr{B}, \mu \times \lambda)$ by setting $S(x, g) = (Tx, \sigma(x)g)$. Suppressing G, we denote this system by (S, T, σ, X) and call it a G-extension of T. The main result we will prove (Theorem 2) is the following:

If (S, T, σ, X) and $(\bar{S}, \bar{T}, \bar{\sigma}, \bar{X})$ are ergodic G-extensions of T and $\bar{T}$, respectively, then they are factor orbit equivalent (relative to the factors $\mathscr{A} \times \{G\}$ and $\bar{\mathscr{A}} \times \{G\}$) via an orbit equivalence $\Phi : X \times G \to \bar{X} \times G$ of the form $\Phi(x, g) = (\phi(x), \alpha(x)g)$ where α is a G-valued measurable function on X.

The method we use to prove this theorem originates with an idea of D. Rudolph. He observed several years ago that a proof of Dye's theorem on the orbit equivalence of ergodic measure-preserving transformations [1] could be given using an argument analogous to that of D. Ornstein in the proof of the isomorphism theorem for B-shifts [4]. At the heart of the argument is a metric on sequences of symbols that plays the role of the $\bar{d}$ metric in [4]. (Already the $\bar{f}$ metric, introduced by J. Feldman [2], had been used in an analogous way by B. Weiss [6] in the proof of the equivalence theorem for finitely fixed transformations.)

2. Preliminaries

In adapting Rudolph's idea to the present situation, we avail ourselves of much of the technique developed in [5]. In particular, we make use of the metric on distributions introduced in that paper, and we begin here by summarizing the relevant facts about this metric. In order to include distributions of countably infinite partitions as a special case, we extend the definition to σ-compact metric spaces. Throughout the following discussion, (M, ρ) will denote a σ-compact metric space with bounded metric ρ.

Definition 1. Let $\{(X_i, \mathscr{B}_i, \mu_i)\}$, $i = 1, 2$ be probability spaces and $f_i : X_i \to (M, \rho)$ Borel measurable functions. A joining of (X_1, f_1) and (X_2, f_2) is a probability space (Z, C, σ) with measure-preserving maps $\Pi_i : Z \to X_i$. We set

$$\left\| \operatorname*{dist}_{X_1} f_1, \operatorname*{dist}_{X_2} f_2 \right\| = \inf_{\{\text{all joinings}\}} \int_Z \rho(f_1\Pi_1(Z), f_2\Pi_2(Z))\, d\sigma(Z).$$

It is easy to see that if X_1 is non-atomic and $\varepsilon > 0$, then there exists a measure-preserving map $\theta : X_1 \to X_2$ such that

$$\int_{X_1} \rho(f_1(x), f_2(\theta x))\, d\mu_1(x) < \left\| \operatorname*{dist}_{X_1} f_1, \operatorname*{dist}_{X_2} f_2 \right\| + \varepsilon.$$

By an (((M,ρ)-valued) process we mean a pair (T,f) consisting of a transformation T on X and a measurable function $f: X \to M$. Given a process (T,f) on X and $x \in X$, $n \in \mathbf{N}$, we let $(1/n)\Sigma_{j=0}^{n-1} \operatorname{dist}_{\{T^jx\}} f$ denote the distribution of f restricted to $\{T^jx\}_{j=0}^{n-1}$ where each $\{T^jx\}$ is given point mass $1/n$. A process (T,f) is ergodic if T restricted to the factor $(f)_T$ generated by the translates of f is ergodic. We will make use of the ergodic theorem and the strong Rokhlin lemma, formulated in this context as follows.

Lemma 1. *If (T_1,f_1) and (T_2,f_2) are ergodic processes on $(X_i,\mathcal{B}_i,\mu_i)$ and $\varepsilon>0$, then $(\exists N_0 \in \mathbf{N})$ $(\forall N \geqq N_0)$ $(\exists A_i \subset X)$ with $\mu(A_i)>1-\varepsilon$ and*

$$(\forall x \in A_i) \quad \left\| \frac{1}{N} \sum_{j=0}^{N-1} \operatorname*{dist}_{\{T_i^jx_i\}} f_i, \operatorname*{dist}_{x_i} f_i \right\| < \varepsilon \quad \textit{and} \quad (\forall x_i \in A_i) \quad i=1,2,$$

there is a bijection $\theta: \{T_1^jx_1\}_{j=0}^{N-1} \to \{T_2^jx_2\}_{j=0}^{N-1}$ such that

$$\frac{1}{N}\sum_{j=0}^{N-1} \rho(f_1(T_1^jx_1), f_2(\theta(T_1^jx_1))) < \left\| \operatorname*{dist}_{x_1} f_1, \operatorname*{dist}_{x_2} f_2 \right\| + \varepsilon.$$

Lemma 2. ***Given an ergodic transformation T on $(X,\mathcal{B},\mu)$, a function** $f: X \to (M,\rho)$ and $\varepsilon>0$, $N \in \mathbf{N}$, there exists a set $F \in \mathcal{B}$ such that $\{T^jF\}_{j=0}^{N-1}$ are disjoint, $\mu(\bigcup_{j=0}^{N-1} T^jF) > 1-\varepsilon$, and $\|\operatorname{dist}_F f, \operatorname{dist}_X f\| < \varepsilon$. (Here the measure on F is $\mu/\mu(F)$.)*

Given functions $f_i: (X_i,\mathcal{B}_i,\mu_i) \to (M,\rho)$, $i=1,2,\cdots,n$, we let $(1/n)\Sigma_{i=0}^{n-1} \operatorname{dist}_{X_i} f_i$ denote the distribution of the disjoint union of the f_i on $(\bigcup_{i=1}^n X_i, (1/n)\Sigma_{i=0}^{n-1}\mu_i)$. Given functions. $f_i: X \to (M_i,\rho_i)$, $i=1,2,\cdots,n$, we let $\bigvee_{i=0}^{n-1} f_i : X \to \Pi_{i=0}^{n-1} M_i$ denote the cartesian product of the f_i, where $\Pi_{i=0}^{n-1} M_i$ is endowed with the sup metric.

3. The $\bar{r}$ metric

The metric on sequences referred to above is given by the following.

Definition 2. Given (M,ρ), $n \in \mathbf{N}$ and $a, b \in M^n$. Set

$$\bar{r}(a,b) = \inf_{\pi \in S_n} \{\bar{d}(\pi a, b) + \|\pi\|\} \qquad \text{where } \bar{d}(\pi a, b) = \frac{1}{n}\sum_{i=1}^n \rho((\pi a)_i, (b)_i)$$

and $\|\pi\| = |\{i \in \{1,2,\cdots,n-1\} : \pi(i)+1 \neq \pi(i+1)\}|$ which we might call the number of cuts in π. One readily verifies that $\bar{r}$ is a metric on M^n.

Given an (M,ρ)-valued process (T,f) on X and $n \in \mathbf{N}$, the n-name of $x \in X$ is the sequence $\{f(T^ix)\}_{i=1}^n \in M^n$. The following lemma establishes a property of ergodic processes which is an analogue of the finitely determined property of the isomorphism theory, and the finitely fixed property of the equivalence theory. Loosely stated, it says that if the finite distributions of two ergodic processes are sufficiently close, then for large n, their n-names are close in the $\bar{r}$ metric.

LEMMA 3. *$(\forall \varepsilon > 0)(\exists \delta > 0, n \in \mathbf{N})$ such that if (T_1, f_1) and (T_2, f_2) are ergodic (M, ρ) valued processes on $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ respectively, such that*

$$(1) \qquad \left| \operatorname{dist}_{X_1} \bigvee_{j=0}^{n-1} T_1^{-j} f_1, \operatorname{dist}_{X_2} \bigvee_{j=0}^{n-1} T_2^{-j} f_2 \right| < \delta.$$

***Then** $(\forall \eta > 0)(\exists N_0 \in \mathbf{N})$ **such that** $(\forall N \geqq N_0)$ **and for** $i = 1, 2$, $(\exists A_i \subset X_i)$, $\mu_i(A_i) > 1 - \eta$, **such that** $(\forall x_1 \in A_1, x_2 \in A_2)$,*

$$(2) \qquad \bar{r}\left(\left(\bigvee_{j=0}^{N-1} T_1^{-j} f_1 \right)(x_1), \left(\bigvee_{j=0}^{N-1} T_2^{-j} f_2 \right)(x_2) \right) < \varepsilon.$$

PROOF. Fix $\varepsilon > 0$ and let $\delta = \varepsilon/100$ and $n > 100/\varepsilon$. Let (T_1, f_1) and (T_2, f_2) be ergodic processes satisfying (1). Fix $\eta > 0$, which we may take to be less than $\varepsilon/100$.

By Lemma 1, $(\exists L \in N)$, $L > n/\eta$, such that for $i = 1, 2$, and sets $B_i \subset X$ with $\mu_i(B_i) > 1 - \eta$, we have, for all $x_1 \in B_1$ and $x_2 \in B_2$,

$$(3) \qquad \left\| \frac{1}{L} \sum_{k=0}^{L-1} \operatorname{dist}_{\{T_1^k x_1\}} \left(\bigvee_{j=0}^{n-1} T_1^{-j} f_1 \right), \frac{1}{L} \sum_{k=0}^{L-1} \operatorname{dist}_{\{T_2^k x_2\}} \left(\bigvee_{j=0}^{n-1} T_2^{-j} f_2 \right) \right\| < \delta,$$

and this distribution match is effected by a bijection

$$\theta : \{T_1^k x_1\}_{k=0}^{L-1} \to \{T_2^k x_2\}_{k=0}^{L-1}.$$

We will refer to the L-names of points in B_i as good L-names. Now let $R = \{R_0, R_1, \cdots, R_s\}$ be a partition of M^L (endowed with the sup metric) such that each R_i, $i = 1, \cdots, s$, has diameter less than η, and $\mu_i(f_i^{-1}(R_0)) < \eta$. If $b \in M^L$, we denote the element of R containing b by $R(b)$.

Again by Lemma 1, $(\exists N_0 \in \mathbf{N})$ such that $(\forall N \geqq N_0)$ and $i = 1, 2$, $(\exists A_i \subset X_i)$ with $\mu_i(A_i) > 1 - \eta$ and $(\forall x \in A_i)$ the N-name a of x contains a disjoint union $\{b_l\}_{l=1}^{M}$ of good L-names which covers all but a fraction 2η of a, and which has the property that each $R(b_l) \in \{R_1, \cdots, R_s\}$ appears at least K times in $\{R(b_l)\}_l$, where $K > 100n/\varepsilon$.

Now fix $x_i \in A_i$, $i = 1, 2$, and let a_1 and a_2 denote their respective N-names. We show that condition (2) holds. Let $\{b_l^i\}_l$ be a set of good L-names in a_i as described above and let $\{b_{l_j}^i\}_{j=1}^{K' > K}$ denote those b_l^i at which $R(b_l^i)$ is a given value.

Subdivide each $b^i_{l_j}$ into consecutive disjoint n-names by making $s < n$ the initial position of the subdivision of $b^i_{l_j}$ whenever $j \equiv s \bmod n$. Repeating this subdivision for each such family $\{b^i_{l_j}\}_j$, we cover a fraction $1-3\eta$ of a_i with a collection $\{c^i_l\}$ of disjoint n-names, and by virtue of the distribution condition (3), there is a bijection $\theta : \{c^1_l\}_l \to \{c^2_l\}_l$ effecting a distribution match to within $\delta + 2\eta$. By permuting a_1 to bring each (c^1_l) into the same position as $\theta(c^1_l)$, we will have obtained the desired $\bar{r}$ match. ■

4. The factor orbit equivalence theorem

Lemma 5 below is the central copying argument that makes the whole proof go. It says, roughly, that if two ergodic G-extensions are sufficiently close in distribution, then a slight modification (in the sense of factor orbit equivalence) will suffice to vastly improve the distribution match. The proof is really a simple application of the property enunciated in Lemma 3. In brief, if the G-extensions are close in distribution, then their names are close in $\bar{r}$. If their names are close in $\bar{r}$, then a slight rearrangement of the orbits followed by a slight relabeling will suffice to vastly improve the distribution match. Regrettably, considerable technical detail is involved in carrying out this argument. In doing so, we will make use of the following notation. If (S, T, σ, X) is a G-extension, then for $n \in \mathbf{Z}$ and $x \in X$, we set

$$\sigma^{(n)}(x) = \begin{cases} \sigma(T^{n-1}x)\sigma(T^{n-2}x)\cdots\sigma(x), & n > 0, \\ \mathrm{id}_G, & n = 0, \\ \sigma(T^n x)^{-1}\sigma(T^{n+1}x)^{-1}\cdots\sigma(T^{-1}x)^{-1}, & n < 0. \end{cases}$$

If T_1 is another transformation on X such that for some function $k : X \to \mathbf{Z}$, and for a.e. $x \in X$, $T_1(x) = T^{k(x)}(x)$, we indicate this by $T_1 = T^{(k)}$. Similarly, we let $\sigma^{(k)}$ denote the function $x \mapsto \sigma^{(k(x))}(x)$. We let $O_T(x)$ denote the T-orbit of x, and for $\varepsilon > 0$, we write $T_1 \sim_\varepsilon T$ if $O_T(x) = O_{T_1}(x)$ a.e. and $\mu\{x \in X : T_1(x) \neq T(x)\} < \varepsilon$. Finally, we denote the second coordinate map on $X \times G$ by c, $c(x, g) = g$. All partitions are assumed to be finite or countably infinite and measurable. It will help if we isolate the following preliminary result.

Lemma 4. *Let (S, T, σ, X) be a G-extension, P a partition on X and let $(x_0, g_0) \in X \times G$ satisfy*

$$\left\| \frac{1}{k} \sum_{i=0}^{k-1} \operatorname*{dist}_{\{S^i(x_0, g_0)\}} \left(\bigvee_{j=0}^{n-1} S^{-j}(P \vee c) \right), \operatorname*{dist}_{X \times G} \left(\bigvee_{j=0}^{n-1} S^{-j}(P \vee c) \right) \right\| < \eta$$

for some k and $n \in \mathbf{N}$. Then

$$\left\| \frac{1}{k} \sum_{i=0}^{k-1} \operatorname{dist}_{S^i(\{x_0\}\times G)} \left(\bigvee_{j=0}^{n-1} S^{-j}(P \vee c) \right), \operatorname{dist}_{X\times G} \left(\bigvee_{j=0}^{n-1} S^{-j}(P \vee c) \right) \right\| < \eta.$$

(The measure on each $S^i(\{x_0\}\times G)$ is $\delta_{\{x_0\}} \times \lambda$.)

PROOF. Let $\theta : X \times G \to \{S^i(x_0, g_0)\}_{i=0}^{k-1}$ be a measure-preserving map such that

$$\int_{X\times G} m\left(\bigvee_{j=0}^{n-1} S^j(P \vee c)(x,g), \bigvee_{j=0}^{n-1} S^{-j}(P \vee c)(\theta(x,g)) \right) d\nu < \eta$$

where m is the appropriate metric. Let $Z = (X \times G) \times G$, with product measure $\nu \times \lambda$ and let $\pi_1 : Z \to X \times G$ via $\pi_1(x, g_1, g_2) = (x, g_1 g_2)$ and $\pi_2 : Z \to \{S^i(\{x_0\} \times G)\}_{i=0}^{k-1}$ via $\pi_2((x, g_1), g_2) = T_{g_2}(\theta(x, g_1))$ where $T_{g_2}(x, g) = (x, gg_2)$. Then

$$\int_Z m\left[\bigvee_{j=0}^{n-1} S^{-j}(P \vee c)(\pi_1(x, g_1, g_2)), \bigvee_{j=0}^{n-1} S^{-j}(P \vee c)(\pi_2(x, g_1, g_2)) \right] d\nu \times \lambda\,(x, g_1, g_2)$$

$$= \int_Z m\left[\bigvee_{j=0}^{n-1} S^{-j}(P \vee c)(x, g_1 g_2), \bigvee_{j=0}^{n-1} S^{-j}(P \vee c)(T_{g_2}(\theta(x, g_1))) \right] d\nu \times \lambda\,(x, g_1, g_2)$$

$$= \int_Z m\left[\bigvee_{j=0}^{n-1} S^{-j}(P \vee c)(x, g_1), \bigvee_{j=0}^{n-1} S^{-j}(P \vee c)(\theta(x, g_1)) \right] d\nu \times \lambda\,(x, g_1, g_2) < \eta.$$

■

LEMMA 5. *$(\forall \varepsilon > 0)(\exists \delta > 0, n \in \mathbf{N})$ such that if (S, T, σ, X) and $(\bar S, \bar T, \bar\sigma, \bar X)$ are ergodic G-extensions with partitions P of X and $\bar P$ of X such that*

$$\left\| \operatorname{dist}_{X\times G} \bigvee_{i=0}^{n-1} S^{-i}(P \vee c), \operatorname{dist}_{X\times G} \bigvee_{i=0}^{n-1} \bar S^{-i}(\bar P \vee c) \right\| < \delta \tag{4}$$

then $(\exists N \in \mathbf{N})$ such that $(\forall \delta_1 > 0, n_1 \in \mathbf{N})$ there exists a measurable $\alpha : \bar X \to G$ and $\bar P_1$ on $\bar X$ and $\bar T_1 = \bar T^{k(x)}$ such that

$$\int_{\bar X} \rho(\alpha(\bar x), \mathrm{id}_G)\, d\bar\mu < \varepsilon, \tag{5}$$

$$|\bar P_1 - \bar P| < \varepsilon, \tag{6}$$

$$\bar T_1 \underset{\varepsilon}{\sim} \bar T, \tag{7}$$

$$\text{for a.e. } \bar x \in \bar X, \quad \bar x = \bar T^m \bar x' \text{ and } \bar x = \bar T_1^{m_1} \bar x' \text{ imply } |m - m_1| < 2N, \tag{8}$$

$$\left\| \operatorname{dist}_{X\times G} \bigvee_{i=0}^{n_1-1} S^{-i}(P \vee c), \operatorname{dist}_{X\times G} \bigvee_{i=0}^{n_1-1} \bar S^{-i}(\bar P_1 \vee \alpha c) \right\| < \delta_1 \quad \text{where } \bar S_1 = \bar S^{(k)}. \tag{9}$$

PROOF. Fix $\varepsilon > 0$. Choose (δ, n) by Lemma 3 for an $\bar{r}$ match to $\varepsilon/100$. Let (S, T, σ, X) and $(\bar{S}, \bar{T}, \bar{\sigma}, \bar{X})$ be ergodic G-extensions satisfying (4) for this choice of (δ, n). Again by Lemma 3, choose $N \in \mathbf{N}$ so that for sets $A \subset X \times G$ and $\bar{A} \subset \bar{X} \times G$ of measure greater than $1 - \varepsilon/100$, and for all $(x, g) \in A$ and $(\bar{x}, \bar{g}) \in \bar{A}$ we have $\bar{r}(\vee_{j=1}^{N} S^{-j}(P \vee c)(x, g), \vee_{j=1}^{N} \bar{S}^{-j}(\bar{P} \vee c)(\bar{x}, \bar{g})) < \varepsilon/100$. We must show that these choices of (δ, n) and N have the desired properties. Fix $\delta_1 > 0$ and $n_1 \in \mathbf{N}$ (we may assume $n_1 > N$) and choose $\eta \in (0, 1)$, whose size will be determined by subsequent considerations. Let $K > n_1/\eta$ be an integer multiple of N such that for set $\bar{B} \subset \bar{X} \times G$ of measure greater than $1 - \eta$, and for all $(\bar{x}, g) \in B$, $|\{j \in \{0, \cdots, K-1\} : \bar{S}^j(\bar{x}, g) \in \bar{A}\}| > (1 - \eta)K$ and such that there exists a point $(x_1, g_1) \in X \times G$ with

$$\left\| \frac{1}{K} \sum_{j=0}^{K-1} \operatorname*{dist}_{S^j(x_1, g_1)} \left(\bigvee_{i=0}^{n-1} S^{-i}(P \vee c) \right), \operatorname*{dist}_{X \times G} \left(\bigvee_{i=0}^{n-1} S^{-i}(P \vee c) \right) \right\|$$

and

$$|\{j \in \{0, \cdots, K-1\} : S^j(x_1, g_1) \in A\}| > (1 - \eta)K.$$

Let $\bar{Q} \supset \bar{P}$ be a generator for $\bar{T}$. Let $M > K/\eta$ be an integer multiple of K such that for a collection $\mathscr{E} \subset \vee_{i=0}^{M} \bar{S}^{-i}\bar{Q}$ with $\bar{\mu}(\bigcup \mathscr{E}) > 1 - \eta$, and $(\forall E \in \mathscr{E})(\exists c_{ljE} \in G)$, $l = 1, \cdots, M$, $j = 1, \cdots, K$ such that

$$\frac{1}{M} \sum_{l=0}^{M-1} \int_{\bar{S}^l E} \sum_{j=0}^{K-1} \rho(\bar{\sigma}_j(\bar{x}), c_{ljE}) d\bar{\mu}(\bar{x}) < \eta \bar{\mu}(E).$$

Now construct a Rokhlin tower in $(\bar{X}, \bar{T})$ of height M, with base F and $\bar{\mu}(\bigcup_{j=0}^{M-1} \bar{T}^j F) > 1 - \eta$ such that

$$\left\| \operatorname*{dist}_{F} \bigvee_{j=0}^{M-1} \bar{T}^{-j} \left(\bar{Q} \vee \bigvee_{i=0}^{K-1} \bar{\sigma}_i \right), \operatorname*{dist}_{\bar{X}} \bigvee_{j=0}^{M-1} \bar{T}^{-j} \left(\bar{Q} \vee \bigvee_{i=0}^{K-1} \bar{\sigma}_i \right) \right\| < \eta_1,$$

where η_1 is chosen so small that $\bar{\mu}(F \cap \bigcup \mathscr{E}) > (1 - \eta)\bar{\mu}(F)$ and $(\forall E \in \mathscr{E})$

$$\frac{1}{M} \sum_{l=0}^{M-1} \int_{\bar{T}^l(E \cap F)} \sum_{j=0}^{K-1} \rho(\bar{\sigma}_j(\bar{x}), c_{ljE}) d\bar{\mu}(\bar{x}) < \eta \bar{\mu}(E \cap F).$$

Let $\bar{C} = \{\bar{x} \in \bar{X} : \lambda(\{g \in G : (\bar{x}, g) \in \bar{B}\}) > 1 - \sqrt{\eta}\}$. Then $\bar{\mu}(\bar{C}) > 1 - \sqrt{\eta}$ so $\bar{\mu}(\bar{C} \cap \bigcup_{j=0}^{M-1} \bar{T}^j F) > (1 - \sqrt{4\eta})\bar{\mu}(\bigcup_{j=0}^{M-1} \bar{T}^j F)$, so for a collection $\mathscr{E}' \subset \mathscr{E}$ with $\bar{\mu}((\bigcup \mathscr{E}') \cap F) > (1 - \sqrt[4]{4\eta})\bar{\mu}((\bigcup \mathscr{E}) \cap F)$ and $(\forall E' \in \mathscr{E}')$ we have

$$\bar{\mu}\left(\bar{C} \cap \bigcup_{j=0}^{M-1} \bar{T}^j(E' \cap F) \right) > (1 - \sqrt[4]{4\eta})\bar{\mu}\left(\bigcup_{j=0}^{M-1} \bar{T}^j(E' \cap F) \right).$$

Fix such an E'. Then for a set of integers $L \in \{0, \cdots, K-1\}$ of density greater than $1 - \sqrt[8]{4\eta}$,

$$\bar{\mu}\left(\bar{C}\cap\bigcup_{j=0}^{M/K-1}\bar{T}^{L+jK}(E'\cap F)\right)>(1-\sqrt[8]{4\eta})\bar{\mu}\left(\bigcup_{j=0}^{M/K-1}\bar{T}^{L+jK}(E'\cap F)\right).$$

Furthermore, for a set of integers $L\in\{0,\cdots,K-1\}$ of density greater than $1-\sqrt{\eta}$, we have

$$\frac{K}{M}\sum_{j=1}^{M/K-1}\int_{\bar{T}^{L+jK}(E'\cap F)}\sum_{i=0}^{K-1}\rho(\bar{\sigma}_i(\bar{x}),c_{L+jK,i,E})d\bar{\mu}(\bar{x})<\sqrt{\eta}\bar{\mu}(E'\cap F).$$

Thus, if η is sufficiently small, $\exists L\in\{0,\cdots,K-1\}$ for which both these conditions hold. For such an L, there is a set of integers $J\subset\{0,\cdots,M/K-1\}$ of density greater than $1-\sqrt[16]{4\eta}-\sqrt[4]{\eta}$ such that $(\forall j\in J)$

$$\bar{\mu}(\bar{C}\cap\bar{T}^{L+jK}(E'\cap F))>(1-\sqrt[16]{4\eta})\bar{\mu}(E'\cap F)\text{ and }\sum_{i=0}^{K-1}\rho(\sigma_i(\bar{x}),\ c_{L+jK,i,E'})<\sqrt[8]{\eta}$$

for $\bar{x}$ in a set $\bar{C}_j\subset\bar{T}^{L+jK}(E'\cap F)$ of measure greater than $(1-\sqrt[8]{\eta})\mu(E'\cap F)$. Hence (if η is sufficiently small), we may choose $\bar{x}_1\in\bar{C}\cap\bar{C}_j$ and $\bar{g}_1\in G$ so that $(\bar{x}_1,\bar{g}_1)\in\bar{B}$. Now choose $l\in\{0,\cdots,N-1\}$ so that except for a set of $m\in\{0,\cdots,K/N\}$ of density less than 2η, the $(P\vee c)$ (respectively $\bar{P}\vee c$) N-names of the points $S^{l+mN}(X_1,g_1)$ (respectively, $S^{l+mN}(\bar{x}_1,\bar{g}_1)$) are within $\varepsilon/100$ of each other in the $\bar{r}$-metric. Let $\pi\in S_K$ permute the column levels $\{\bar{T}^i(E'\cap F)\}_{i=L+jK}^{L+(j+1)K-1}$ within each of the segments of length N corresponding to the names just described, in such a way as to effect the $\bar{r}$ matches there. Let $\bar{T}_1=\bar{T}^{(k)}$ denote the transformation obtained on (this portion of) $\bar{X}$ by translating these column levels in their new order, and $(\bar{S}_1,\bar{T}_1,\bar{\sigma}_1,\bar{X})$ the corresponding G-extension, where $\bar{\sigma}_1=\bar{\sigma}^{(k)}$ (and $\bar{S}_1=\bar{S}^{(k)}$).

Let $(\bar{x}_2,g_2)$ denote the point in the $\bar{S}$ orbit of $(\bar{x}_1,g_1)$ that now occupies the initial position in this segment of the orbit of $\bar{S}_1$, and let D denote the translate of $E'\cap F$ that contains $(\bar{x}_2,g_2)$. It will be convenient to let $C_{i,D}$ denote $C_{L+jK,\pi^{-1}(i),E'}$, for $i\in\{0,\cdots,K-1\}$.

On $\bigcup_{i=0}^{K-1}\bar{T}_1^iD$ we define $\bar{P}_1$ and α as follows: $(\forall\bar{x}\in D)$ and $i\in\{0,1,\cdots,K-1\}$. $\bar{P}_1(\bar{T}_1^i\bar{x})=P(T^ix_1)$ and

$$\alpha(\bar{T}_1^i\bar{x})=\sigma_i(x_1)g_1g_2^{-1}\bar{\sigma}_{1i}(\bar{x})^{-1}.$$

We repeat this construction for each $j\in J$ and each $E'\in\mathscr{E}'$, and set $\alpha(\bar{x})=\mathrm{id}_G$ and $\bar{P}_1(\bar{x})=\bar{P}(\bar{x})$ on the remainder of $\bar{X}$. We now can see that conditions (5) through (9) have been satisfied. (5): Let D, (x_1,g_1) and $(\bar{x}_2,g_2)$ be as above. Then

$$\sum_{i=0}^{K-1}\int_D \rho(\alpha(\bar{T}_1^i(\bar{x})),\mathrm{id}_G)d\bar{\mu}(\bar{x})=\sum_{i=0}^{K-1}\int_D \rho(\sigma_i(x_1)g_1,\bar{\sigma}_{1i}(\bar{x})g_2)d\bar{\mu}(\bar{x})$$

$$\leqq \sum_{i=0}^{K-1}\int_D [\rho(\sigma_i(x_1)g_1,\bar{\sigma}_{1i}(\bar{x}_2)g_2)+\rho(\bar{\sigma}_{1i}(\bar{x})g_2,c_{iD}g_2)+\rho(c_{iD}g_2,\bar{\sigma}_{1i}(\bar{x})g_2)]d\bar{\mu}(\bar{x})$$

$$<(\varepsilon/100+3\eta)K\bar{\mu}(D)+\sqrt[8]{\eta}\,\bar{\mu}(D)+\sqrt[4]{\eta}\,\bar{\mu}(D).$$

Thus, for sufficiently small η,

$$\int_{\bigcup_{i=0}^{K-1}\bar{T}_1^iD}\rho(\alpha(\bar{x}),\mathrm{id}_G)d\bar{\mu}(\bar{x})<\frac{\varepsilon}{50}\bar{\mu}\left(\bigcup_{i=0}^{K-1}\bar{T}_1^iD\right).$$

The sets of the form $\bigcup_{i=0}^{K-1}\bar{T}_1^iD$ cover a set of measure $1-5\sqrt[16]{\eta}$, so for sufficiently small η, we obtain (5).

Because of the $\bar{f}$-match used to construct $\bar{P}_1$ and $\bar{T}_1$, the density of the changes made in $\bar{P}$ and $\bar{T}$ along the columns of the tower is less than $\varepsilon/100+3\eta$, which gives (6) and (7).

Condition (8) is clear. To verify condition (9), again let D, (x_1,g_1) and $(\bar{x}_2,g_2)$ be as above. Consideration of the map $\theta:\bigcup_{i=0}^{K-1}\bar{T}_1^iD\to\bigcup_{i=0}^{K-1}T^i(\{x_1\}\times G)$ given by $\bar{T}_1^i(\bar{x},g_2g)\to T^i(x_1,g_1g)$, informs us that

$$\left\|\operatorname*{dist}_{\bigcup_{i=0}^{K-1}\bar{T}_1^iD}\bigvee_{j=0}^{n_1-1}\bar{S}_1^j(\bar{P}_1\vee\alpha c),\ \operatorname*{dist}_{\bigcup_{i=0}^{K-1}T^i(\{x_1\}\times G)}\bigvee_{j=0}^{n_1-1}S^j(P\vee c)\right\|<\eta$$

and

$$\left\|\operatorname*{dist}_{\bigcup_{i=0}^{K-1}T^i(\{x_1\}\times G)}\bigvee_{j=0}^{n_1-1}S^{-j}(P\vee c),\ \operatorname*{dist}_{X\times G}\bigvee_{j=0}^{n_1-1}S^{-j}(P\vee c)\right\|<\eta,$$

since $n_1/K<\eta$ and by Lemma 4. But the measure of the union of the sets of the form $\bigcup_{i=0}^{K-1}\bar{T}_1^iD$ is (for sufficiently small η) large enough for these distribution conditions to imply (9). ■

Theorem 1, below, is the analogue in the present context of the "strong Sinai" theorem in the theory of Bernoulli shifts (cf. [4]). This analogy will be made explicit as a corollary to the theorem.

We will use the following simple lemma, whose proof is omitted.

LEMMA 6. $(\forall N\in\mathbf{N},\eta>0)$ *if* $0<\varepsilon<\eta/2N$ *and* T_1 *and* T_2 *are transformations on* $(X,\mathcal{B},\mu)$ *such that* $T_1\sim_\varepsilon T_2$ *then*

$$\mu\{x\in X: T_1^ix=T_2^ix \text{ for } i\in\{-N,\cdots,N\}\}>1-\eta.$$

THEOREM 1. $(\forall\varepsilon>0)(\exists\delta>0$ *and* $n\in\mathbf{N})$ *such that if* (S,T,σ,X) *and* $(\bar{S},\bar{T},\bar{\sigma},\bar{X})$ *are ergodic* G*-extensions with partitions* P *of* X *and* $\bar{P}$ *of* $\bar{X}$ *satisfying*

(10) $$\left\| \operatorname*{dist}_{X\times G} \bigvee_{i=0}^{n-1} S^{-i}(P \vee c), \operatorname*{dist}_{X\times G} \bigvee_{i=0}^{n-1} \bar{S}^{-i}(\bar{P} \vee c) \right\| < \delta$$

then there exists a transformation $\hat{T}$, a partition $\hat{P}$ on $\bar{X}$, and a measurable function $\hat{\alpha} : \bar{X} \to G$ such that

(11) $$\int_{\bar{X}} \rho(\hat{\alpha}(x), \mathrm{id}_G) d\bar{\mu}(\bar{x}) < \varepsilon,$$

(12) $$|\hat{P} - \bar{P}| < \varepsilon,$$

(13) $$\hat{T} \underset{\varepsilon}{\sim} \bar{T},$$

(14) *if $k : \bar{X} \to \mathbf{Z}$ is such that $\hat{T} = \bar{T}^{(k)}$ and $\hat{S} = \bar{S}^{(k)}$ then* $(\forall m \in \mathbf{N})$

$$\left\| \operatorname*{dist}_{X\times G} \bigvee_{i=0}^{m} S^{-i}(P \vee c), \operatorname*{dist}_{\bar{X}\times G} \bigvee_{i=0}^{m} \hat{S}^{-i}(P \vee c) \right\| = 0.$$

Proof. Fix $\varepsilon > 0$ and a sequence $\{\varepsilon_i\}_{i=1}^{\infty}$, $\varepsilon_i > 0$, such that $\sum_{i=1}^{\infty} \varepsilon_i < \varepsilon$. Choose (δ, n) as in Lemma 5 for ε_1, and suppose that (10) holds for this (δ, n). Let N_1 be the corresponding integer given by Lemma 5. Let $\varepsilon_2' \in (0, \varepsilon_2)$ satisfy $\varepsilon_2' < 1/4N_1$ and choose (δ_1, n_1) by Lemma 5 with respect to ε_2'. Then Lemma 5 gives us $\bar{T}_1 = \bar{T}^{(k_1)}$, $\bar{P}_1$, and α_1 on $\bar{X}$ satisfying conditions (5) through (9).

Let $\bar{\sigma}_1 = (\alpha_1 \cdot \bar{T}_1)\bar{\sigma}^{(k_1)}(\alpha_1)^{-1}$ and let $\bar{S}_1$ be the G-extension determined by $\bar{T}_1$ and $\bar{\sigma}_1$. Then condition (9) may be rewritten as

$$\left\| \operatorname*{dist}_{X\times G} \bigvee_{j=0}^{n_1-1} S^{-j}(P \vee c), \operatorname*{dist}_{X\times G} \bigvee_{j=0}^{n_1-1} \bar{S}_1^{-j}(\bar{P}_1 \vee c) \right\| < \delta_1.$$

Let N_2 be the corresponding integer given by Lemma 5. By continuing in the above manner, we obtain a sequence of processes $\{(\bar{T}_i, \bar{P}_i)\}_{i=1}^{\infty}$ on $\bar{X}$, functions $\alpha_i : \bar{X} \to G$, positive integers N_i and $\varepsilon_i' \in (0, \varepsilon_i)$ such that

(15) $\bar{T}_i \underset{\varepsilon_i'}{\sim} \bar{T}_{i-1}$, $|\bar{P}_i - \bar{P}_{i-1}| < \varepsilon_i'$ and $\int_{\bar{X}} \rho(\alpha_i(\bar{x}), \mathrm{id}_G) d\bar{\mu} < \varepsilon_i'$ (where $\bar{T}_0 = \bar{T}$);

(16) if $k_i : \bar{X} \to \mathbf{Z}$ is such that $\bar{T}_i = \bar{T}^{(k_i)}$ then $|k_i(\bar{x})| \leqq \sum_{j=1}^{i} N_j$ (a.e. $\bar{x}$);

(17) $$\varepsilon_i' < 2^{-(i+1)} \left(\sum_{j=1}^{i-1} N_j \right)^{-1};$$

(18) $$\left\| \operatorname*{dist}_{X\times G} \bigvee_{j=0}^{n_i-1} S^{-j}(P \vee c), \operatorname*{dist}_{X\times G} = \bigvee_{j=0}^{n_i-1} \bar{S}_i^{-j}(\bar{P}_i \vee \beta_i c) \right\| < \delta_i$$

where $\beta_i(\bar{x}) = \Pi_{l=1}^{i} \alpha_l(\bar{x})$, $\bar{S}_i = \bar{S}^{(k_i)}$ and $\delta_i < \frac{1}{2}\delta_{i-1}$, $n_i > n_{i-1}$ are chosen as in Lemma 5 with respect to ε'_i.

To complete the proof, we set $\hat{P} = \lim_{i\to\infty} \bar{P}_i$, $\hat{\alpha} = \lim_{i\to\infty} \beta_i$ and $\hat{T}(\bar{x}) = \lim_{i\to\infty} \bar{T}_i(\bar{x})$. (Note that the sequence $\{\bar{T}_i(\bar{x})\}_i$ is eventually constant for a.e. $\bar{x}$.)

Conditions (11), (12) and (14) are then immediately verified. It is clear that $\bar{\mu}\{\bar{x} : T\bar{x} \neq \bar{T}\bar{x}\} < \varepsilon$ and that $O_{\hat{T}}(\bar{x}) \subset O_{\bar{T}}(\bar{x})$ for a.e. $\bar{x}$. Furthermore,

$$\bar{\mu}\left(\bigcap_{j=0}^{\infty} \bigcup_{i=j}^{\infty} \{\bar{x} : [\bar{x}, \bar{T}\bar{x}]_{\bar{T}_i} \neq [\bar{x}, \bar{T}\bar{x}]_{\bar{T}_{i+1}}\}\right) = 0$$

where $[\bar{x}, \bar{T}\bar{x}]_{\bar{T}_i}$ denotes the ordered segement of $O_{\bar{T}_i}(\bar{x})$ between $\bar{x}$ and $\bar{T}\bar{x}$. Hence $\bar{T}(\bar{x}) \in O_{\hat{T}}(\bar{x})$ for a.e. $\bar{x}$, so that $O_{\hat{T}}(\bar{x}) = O_{\bar{T}}(\bar{x})$ and (13) is established. ■

COROLLARY. *Let (S, T, σ, X) and $(\bar{S}, \bar{T}, \bar{\sigma}, \bar{X})$ be ergodic G-extensions. Then $(\forall \varepsilon > 0)$ there exists a G-extension $(\hat{S}, \hat{T}, \hat{\sigma}, \bar{X})$, and functions $k : \bar{X} \to \mathbf{Z}$ and $\alpha : \bar{X} \to G$ such that*

$\hat{T} = \bar{T}^{(k)}$ and $\hat{\sigma} = \bar{\sigma}^{(k)}$,

$\hat{T} \sim_\varepsilon \bar{T}$ and

(S, T, σ, X) is a factor of $(\hat{S}, \hat{T}, \hat{\sigma}, \bar{X})$ via a homomorphism Φ of the form $\Phi(\bar{x}, g) = (\phi(\bar{x}), \alpha(\bar{x})g)$.

PROOF. Fix $\varepsilon > 0$ and choose (δ, n) for ε as in Theorem 1. Let P be a generator for (T, X). Construct a partition $\bar{P}$ on $\bar{X}$ and $\alpha_1 : \bar{X} \to G$ such that

$$\left\| \operatorname*{dist}_{X\times G} \bigvee_{i=0}^{n-1} S^{-i}(P \vee c), \operatorname*{dist}_{\bar{X}\times G} \bigvee_{i=0}^{n-1} \bar{S}^{-i}(\bar{P} \vee \alpha_1 c) \right\| < \delta.$$

Apply Theorem 1 to (S, T, P, σ, X) and $(\bar{S}, \bar{T}, \bar{P}, \bar{\sigma}_1, \bar{X})$ where $\bar{\sigma}_1 = (\alpha(\bar{T}))\bar{\sigma}(\alpha)^{-1}$, to obtain $\hat{T} = \bar{T}^{(k)}$, P, and α_2 satisfying conditions (11) through (14). Let $\alpha = \alpha_2\alpha_1$. Then the distribution condition (14) implies that the map $(\bar{x}, g) \mapsto (\phi(\bar{x}), \alpha(\bar{x})g)$ is a homomorphism, where $\phi : \bar{X} \to X$ is the map that carries $(\bar{T}, \bar{P})$ to (T, P). ■

We are now in a position to obtain the factor orbit equivalence theorem for G-extensions from Theorem 1 by imitating the corresponding portions of the proof of the equivalence theorem [6]. We have made an attempt to preserve the notation of [6] in order to facilitate comparison with that work. One difference between the two arguments that should be noted is the attention we must pay (as in the proof of Theorem 1) to keeping our modifications within the given orbit equivalence class.

LEMMA 7. *Given ergodic G-extensions (S, T, σ, X) and $(\bar{S}, \bar{T}, \bar{\sigma}, \bar{X})$, partitions B of $\bar{X}$ and Q of X, a homomorphism Φ from $\bar{S}$ to S of the form $\Phi(\bar{x}, g) =$*

$(\phi(\bar{x}), \bar{\alpha}(\bar{x})g)$ and $\varepsilon > 0$, there exist $\bar{\alpha}_1 : \bar{X} \to G$ and $\bar{T}_1 = \bar{T}^{(\bar{k})}$, $\alpha_1 : X \to G$ and $T_1 = T^{(k)}$ on X, a homomorphism ϕ_1 from $(\bar{T}_1, \bar{X})$ to (T_1, X) and a generator $Q_1 \supset Q$ for (T_1, X) such that

$$(19) \qquad \bar{T}_1 \underset{\varepsilon}{\sim} \bar{T} \quad \text{and} \quad \int_{\bar{X}} \rho(\bar{\alpha}_1(\bar{x}), \mathrm{id}_G)d\bar{\mu} < \varepsilon,$$

$$(20) \qquad T_1 \underset{\varepsilon}{\sim} T \quad \text{and} \quad \int_X \rho(\alpha_1(x), \mathrm{id}_G)d\bar{\mu} < \varepsilon,$$

and $\Phi_1 : (\bar{x}, g) \to (\phi_1(\bar{x}), (\alpha(\phi_1\bar{x}))^{-1}\bar{\alpha}_1(\bar{x})\bar{\alpha}(\bar{x})g)$ is a homomorphism from $\bar{S}^{(\bar{k})}$ to $S^{(k)}$ such that

$$(21) \qquad |\phi_1^{-1}(Q) - \phi^{-1}(Q)| < \varepsilon \quad \text{and}$$

$$(22) \qquad B \underset{\varepsilon}{\subset} (\phi_1^{-1}(Q_1))_{\bar{T}_1}.$$

PROOF. Fix $\varepsilon > 0$. Let $\bar{P} \supset \bar{B}$ be a generator for $(\bar{T}, \bar{X})$, choose $N_0 \in \mathbf{N}$ so that $\exists \bar{R} \subset \bigvee_{i=-N_0}^{N_0} \bar{T}^{-i}\bar{P}$ with $|\phi^{-1}(Q) - \bar{R}| < \varepsilon/10$.

Let $\varepsilon_1 \in (0, \varepsilon)$, with size to be determined later, and pick (δ_1, n_1) as in Theorem 1 for ε_1. Construct a partition $\tilde{P}$ on X such that

$$\left\| \underset{X \times G}{\mathrm{dist}} \bigvee_{i=0}^{n_1 - 1} \bar{S}^{-i}(\bar{P} \vee \phi^{-1}(Q) \vee \bar{\alpha}c), \underset{X \times G}{\mathrm{dist}} \bigvee_{i=0}^{n_1 - 1} S^{-i}(\tilde{P} \vee Q \vee c) \right\| < \delta_1.$$

Then there exist $T_1 = T^{(k)}$, $\alpha_1 : X \to G$, and a partition P_1 on X satisfying $T_1 \sim_{\varepsilon_1} T$, $\int_X \rho(\alpha_1(x), \mathrm{id}_G)d\bar{\mu} < \varepsilon_1$, $|P_1 - \tilde{P}| < \varepsilon_1$, and

$$(\forall n \in \mathbf{N}) \quad \left\| \underset{X \times G}{\mathrm{dist}} \bigvee_{i=0}^{n} \bar{S}^{-i}(\bar{P} \vee \bar{\alpha}c), \underset{X \times G}{\mathrm{dist}} \bigvee_{i=0}^{n} S_1^{-i}(P_1 \vee \alpha_1 c) \right\| = 0, \quad \text{where } S_1 = S^{(k)}.$$

Let R be the partition formed from atoms of $\bigvee_{i=-N_0}^{N_0} T_1^{-i}(P)$ as $\bar{R}$ is from $\bigvee_{i=-N_0}^{N_0} \bar{T}^{-i}(\bar{P})$.

If ε_1 and δ_1 are sufficiently small, and n_1 is sufficiently large, then $|Q - R| < 2\varepsilon/10$.

Now let $Q_1 \supset P_1 \vee Q$ be a generator for (T_1, X) and choose N_1 so that $P_1 | \subset_{\varepsilon/10} \bigvee_{i=-N_1}^{N_1} T_1^{-i}(P_1)$. Choose $\varepsilon_2 > 0$ (of size to be specified later) and (δ_2, n_2) by Theorem 1 with respect to ε_2. As above, construct a partition $\tilde{Q}$ in $\bar{X}$ so that

$$\left\| \underset{X \times G}{\mathrm{dist}} \bigvee_{i=0}^{n_2 - 1} \bar{S}^{-i}(\bar{P} \vee \tilde{Q} \vee \bar{\alpha}c), \underset{X \times G}{\mathrm{dist}} \bigvee_{i=0}^{n_2 - 1} S_1^{-i}(P_1 \vee Q_1 \vee \alpha_1 c) \right\| < \delta_2$$

and apply Theorem 1 to obtain $\bar{T}_1$, $\bar{\alpha}_1$, and $\bar{Q}_1$ on $\bar{X}$ such that $\bar{T}_1 \sim_{\varepsilon_2} \bar{T}$, $\int_{\bar{X}} \rho(\bar{\alpha}_1(\bar{x}), \mathrm{id}_G)d\bar{\mu} < \varepsilon_2$, $|\bar{Q}_1 - \tilde{Q}| < \varepsilon_2$ and

$$(23)\qquad (\forall n\in\mathbf{N})\quad \left\| \operatorname*{dist}_{\bar X\times G}\bigvee_{i=0}^{n}\bar S_1^{-i}(\bar Q_1\vee\bar\alpha_1\bar\alpha c),\ \operatorname*{dist}_{X\times G}\bigvee_{i=0}^{n}S_1^{-i}(Q_1\vee\alpha_1 c)\right\| = 0.$$

Let ϕ be the homomorphism from $(\bar T_1,\bar X)$ to (T_1,X) carrying $\bar Q_1$ onto Q_1. Then the distribution condition (23) implies that Φ_1 is a homomorphism.

Furthermore, ε_2 and (δ_2,n_2) may have been chosen to insure that $\bar P\subset_{2\varepsilon/10}\bigvee_{i=-N_1}^{N_1}\bar T_1^{-i}(\bar Q_1)$, giving (22), and $|\phi_1^{-1}(Q)-\bar R_1|<3\varepsilon/10$, where R_1 is formed from $\bigvee_{i=-N_0}^{N_0}\bar T_1^{-i}(\bar P)$ as $\bar R$ is from $\bigvee_{i=-N_0}^{N_0}\bar T^{-i}(P)$, while $|\phi^{-1}(Q)-\bar R_1|<2\varepsilon/10$, so that $|\phi_1^{-1}(Q)-\phi^{-1}(Q)|<\varepsilon/2$, giving (21). ■

THEOREM 2. *Given ergodic K-extensions* $(\bar S,\bar T,\bar\sigma,\bar X)$ *and* (S,T,σ,X) *and* $\varepsilon>0$, *there exists* $\bar T'=\bar T^{(\bar k)}\sim_\varepsilon\bar T$ *on* $\bar X$, $T'=T^{(k)}\sim_\varepsilon T$ *on* X *and* $\bar\alpha':\bar X\to G$ *such that if* $\sigma'=\sigma^{(k)}$ *and* $\bar\sigma'=\bar\sigma^{(\bar k)}$ *then* $(\bar S',\bar T',\bar\sigma',\bar X)$ *and* (S',T',σ',X') *are isomorphic via an isomorphism* Φ *of the form* $\Phi(\bar x,g)=(\phi'(\bar x),\bar\alpha'(\bar x)g)$.

Note that Φ is then a factor orbit equivalence between $(\bar S,\bar T,\bar\sigma,\bar X)$ and (S,T,σ,X) of the form described in the introduction.

PROOF. Fix $\{\varepsilon_i\}_{i=0}^{\infty}$ such that $\sum_{i=0}^{\infty}\varepsilon_i<\varepsilon$. By the corollary to Theorem 1, we obtain $\bar T_0=\bar T^{(\bar k_0)}\sim_{\varepsilon_0}\bar T$ on $\bar X$ and $\bar\alpha_1:\bar X\to G$ such that (S,T,σ,X) is a factor of $(\bar S_0,\bar T_0,\bar\sigma_0=\bar\sigma^{(\bar k_0)},\bar X)$ via a homomorphism $(\bar x,g)\to(\phi_0(\bar x),\bar\alpha_0(\bar x)g)$.

Now fix an increasing sequence of partitions $\{B_i\}_{i=0}^{\infty}$ on $\bar X$ and $\{C_i\}_{i=0}^{\infty}$ on X which generate $\bar{\mathcal A}$ and $\mathcal A$, respectively.

By repeated applications of Lemma 7 we obtain a sequence of transformations $\{T_i=T^{(k_i)}\}_{i=1}^{\infty}$ on X and functions $\alpha_i:X\to G$, a corresponding sequence $\{\bar T_i=\bar T^{(\bar k_i)}\}_{i=1}^{\infty}$ and $\bar\alpha_i$ on $\bar X$, homomorphisms $\phi_i:(\bar T_i,\bar X)\to(T_i,X)$ and generators Q_i for T_i such that $(\forall i)$

$$(24)\qquad Q_i\supset Q_{i-1}\vee c_i,$$

$$(25)\qquad (\phi_i^{-1}(Q_j))_{\bar T_i}\underset{\varepsilon_j+\cdots+\varepsilon_i}{\supset}B_j\qquad\text{for all }1\leqq j\leqq i,$$

$$(26)\qquad |\phi_i^{-1}(Q_{i-1})-\phi_{i-1}^{-1}(Q_{i-1})|<\varepsilon_i,$$

$$(27)\qquad \int_X\rho(\alpha_i,\alpha_{i-1})d\mu<\varepsilon_i,\qquad \int_{\bar X}\rho(\bar\alpha_i,\bar\alpha_{i-1})d\bar\mu<\varepsilon_i,$$

(28) the map $(\bar x,g)\to(\phi_i(\bar x),\alpha_i(\phi_i(\bar x))^{-1}\bar\alpha_i(\bar x)g)$ is a homomorphism from $(\bar S_i,\bar T_i,\bar\sigma^{(\bar k_i)},\bar X)$ to $(S_i,T_i,\sigma^{(k_i)},X)$,

(29) $\mu\{x:T_i^n(x)=T_{i-1}^n(x)\text{ for all }n\in\{-|k_i(x)|,|k_i(x)|\}\}>1-\varepsilon_i$

(and similarly for $\bar T_i$).

For suppose we have completed this construction through $i-1$. Choose $N_i \in \mathbf{N}$ so that $\mu\{x : (\exists n \in \mathbf{Z}), |n| < N_i, Tx = T_{i-1}^n(x)\} > 1 - \varepsilon_i/2$ (and similarly for $\bar{T}_{i-1}$). Apply Lemma 7 to choose $\varepsilon_i' \in (0, \varepsilon_i)$ so that $T_i \sim_{\varepsilon_i'} T_{i-1}$ on X implies $\mu\{x : T_i^n x = T_{i-1}^n x$ for all $n \in \{-N_2, \cdots, N_2\}\} > (1 - \varepsilon_i/2)$ (and similarly for $\bar{T}_i \sim_{\varepsilon_i'} \bar{T}_{i-1}$ on $\bar{X}$) and so that if $\bar{T}_i \sim_{\varepsilon_i'} \bar{T}_{i-1}$ on $\bar{X}$ and R is a partition of $\bar{X}$ with $|R - \phi_{i-1}^{-1}(Q_{i-1})| < \varepsilon_i'$ then $B_j \subset_{\varepsilon_j + \cdots + \varepsilon_i} (R)_{\bar{T}_i}$ for each $1 \leqq j < i$.

Now apply Lemma 7 to obtain transformations $T_i = T^{(k_i)} \sim_{\varepsilon_i'} T_{i-1}$ on X and $\bar{T}_i = \bar{T}^{(\bar{k}_i)} \sim_{\varepsilon_i'} \bar{T}_{i-1}$ on $\bar{X}$, functions $\beta : X \to G$ and $\bar{\beta} : \bar{X} \to G$ with $\int_X \rho(\beta(x), \mathrm{id}_G) d\mu < \varepsilon_i$ and $\int_{\bar{X}} \rho(\bar{\beta}(\bar{x}), \mathrm{id}_G) d\bar{\mu} < \varepsilon_i$, a homomorphism $\phi_i : (\bar{T}_i, \bar{X}) \to (T_i, X)$, and a generator Q_i of T_i such that

$Q_i \supset Q_{i-1} \vee C_i$,

$(\phi_i^{-1}(Q_i))_{\bar{T}_i} \supset_{\varepsilon_i} B_i$,

$|\phi_i^{-1}(Q_{i-1}) - \phi_{i-1}^{-1}(Q_{i-1})| < \varepsilon_i'$,

and $(\bar{x}, g) \mapsto (\phi_i \bar{x}, \alpha_{i-1}(\phi_i \bar{x})^{-1} \beta(\phi_i \bar{x})^{-1} \bar{\beta}(\bar{x}) \bar{\alpha}_{i-1}(\bar{x}) g)$ is a homomorphism from $(\bar{S}_i, \bar{T}_i, \bar{\sigma}^{(\bar{k}_i)}, \bar{X})$ to $(S_i, T_i, \sigma^{(k_i)}, X)$. Setting $\bar{\alpha}_i = \bar{\beta} \bar{\alpha}_{i-1}$ and $\alpha_i = \beta \alpha_{i-1}$ we have obtained the desired conditions. To finish the proof we set $T'(x) = \lim_{i \to \infty} T_i(x)$ and $\alpha'(x) = \lim_{i \to \infty} \alpha_i'(x)$. Condition (29) insures that $\{T_i(x)\}_{i=1}^{\infty}$ is eventually constant, and that T' is orbit equivalent to T. Condition (27) insures that $\lim_{i \to \infty} \alpha_i(x)$ exists a.e. Define $\bar{T}'$ and $\bar{\alpha}'$ analogously.

For $A \in \mathcal{A}$, let $[A]$ denote the element of the measure algebra $\mathcal{M}(\mathcal{A})$ determined by A. By conditions (24), (25) and (26), the map $[\phi']^{-1} : [A] \to \lim_{i \to \infty} [\phi_i^{-1}(A)]$ is a measure-preserving isomorphism from $\mathcal{M}(\mathcal{A})$ to $\mathcal{M}(\bar{\mathcal{A}})$ implemented by a measure-preserving point isomorphism $\phi' : \bar{X} \to X$ carrying $\bar{T}$ to T. Condition (28) and the construction of ϕ' insure that if Q' generates T' and $n \in \mathbf{N}$, then

$$\left\| \operatorname*{dist}_{X \times G} \bigvee_{i=0}^{n} \bar{S}'^{-i}(\phi'^{-1}(Q') \vee \bar{\alpha}' c), \operatorname*{dist}_{X \times G} \bigvee_{i=0}^{n} S'^{-i}(Q' \vee \alpha' c) \right\| = 0.$$

Thus $(\bar{x}, g) \mapsto (\phi'(\bar{x}), \alpha'(\phi'(\bar{x}))^{-1} \bar{\alpha}'(\bar{x}) g)$ is an isomorphism. ∎

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